arXiv:math/0002107v1 [math.CO] 14 Feb 2000

ON THE HOMOTOPY THEORY OF ARRANGEMENTS, II

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Abstract. In "On the homotopy theory of arrangements" published in 1986 the authors gave a comprehensive survey of the subject. This article updates and continues the earlier article, noting some key open problems.

Let M be the complement of a complex arrangement. Our interest here is in the topology, and especially the homotopy theory of M, which turns out to have a rich structure. In the first paper of this name [37], we assembled many of the known results; in this paper we wish to summarize progress in the intervening years, to reiterate a few key unsolved questions, and propose some new problems we find of interest.

In the first section we establish some terminology and notation, and discuss general homotopy classification problems. We introduce the matroid-theoretic terminology that has become more prevalent in the subject in recent years. In this section we also sketch Rybnikov's construction of arrangements with the same matroidal structure but non-isomorphic fundamental groups. In Section 2 we consider some algebraic properties of the fundamental group of the arrangement. Properties of interest include the lower central series, the Chen groups, the rational homotopy theory of the complement, and the cohomology of the group. At the time of our first paper many questions in this area were in flux, so we make a special effort here to clarify the situation. The group cohomology is naturally of interest in the third section as well, which focuses on when or if the complement is aspherical. It is this property which fostered much of the initial interest in arrangements (in the guise of the pure braid space); it is of interest that the determination of when the complement is aspherical is far from settled. Finally, in the fourth section we consider what one might call the topology of the fundamental group. We describe group presentations that have been discovered since the publication of [37], including the recent development of braided wiring diagrams. We also sketch the considerable progress in the study of the Milnor fiber associated with an arrangement.

In 1992 the long-awaited book *Arrangements of Hyperplanes,* by Peter Orlik and Hiroaki Terao appeared, to the delight of all of us working in arrangements. We refer the reader to this text as a general reference on arrangements, and adopt their notation and terminology except where specified. We also mention that perhaps the most interesting development in arrangements in the last ten years involves the deep and fascinating connections with hypergeometric functions. We are pleased to refer the reader to the lecture notes of Orlik and Terao [64] from the 1998 Tokyo meeting for a comprehensive exposition of this material.

Date: November 30, 1998.

1991 *Mathematics Subject Classification.* Primary 57N65; Secondary 52B30, 05B35, 14F35, 14F40, 20F36, 20F55.

Key words and phrases. hyperplane arrangements.

This paper is in final form and no version of it will be submitted for publication elsewhere.

1. Combinatorial and topological structure

One significant change in the study of the homotopy theory of arrangements since the publication of [37] has been the introduction of matroid-theoretic terminology and techniques into the subject. In this section we review this approach and describe progress toward the topological classification of hyperplane complements. Refer to [89, 66] for further details on matroids.

1.1. **The matroid of an arrangement.**

Let $V = \mathbb{C}^\ell$ and let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a central arrangement of hyperplanes in V. For each hyperplane H_i choose a linear form $\alpha_i \in V^*$ with $H_i = \ker(\alpha_i)$. The product $Q(\mathcal{A}) = \prod_{i=1}^n \alpha_i$ is the *defining polynomial* of the arrangement.

The *underlying matroid* $G(\mathcal{A})$ of $\mathcal{A}$ is by definition the collection of subsets of $[n] := \{1, \dots, n\}$ given by

$$G(\mathcal{A}) = \{S \subseteq [n] \mid \{\alpha_i \mid i \in S\} \text{ is linearly dependent}\}.$$

Elements of $G = G(\mathcal{A})$ are called *dependent sets*. Minimal dependent sets are called *circuits*. Independent sets and bases are defined in the obvious way. The *rank* $\operatorname{rk}(S)$ of a set $S \subseteq [n]$ is the size of a maximal independent subset of S. The rank of G (or $\mathcal{A}$) is $\operatorname{rk}([n])$. The *closure* $\overline{S}$ of a set S is defined by

$$\overline{S} = \bigcup \{T \subseteq [n] \mid T \supseteq S \text{ and } \operatorname{rk}(T) = \operatorname{rk}(S)\}.$$

A set S is *closed* if $\overline{S} = S$. Closed sets are also called *flats*. The collection of closed sets, ordered by inclusion, forms a geometric lattice $L(G)$ which is isomorphic to the intersection lattice $L(\mathcal{A})$ defined and studied in [65]. The isomorphism $L(G) \to L(\mathcal{A})$ is given by $S \mapsto \bigcap_{i \in S} H_i$.

Thus the matroid $G(\mathcal{A})$ contains the same information as the intersection lattice $L(\mathcal{A})$. One of the simple advantages of the matroid-theoretic approach is the fact that the matroid $G(\mathcal{A})$ is determined uniquely by any of a number of different pieces of data besides the set of flats. For instance, the set of circuits, the rank function, or the set of bases, each determine the matroid, and thus the intersection lattice. Besides giving a nice conceptual framework for the combinatorial structure of arrangements, techniques and deep results from the matroid theory literature have been applied with some benefit in the study of the topology of arrangements.

The line generated by α_i in V^* depends only on H_i, and thus $\mathcal{A}$ determines a unique point configuration $\mathcal{A}^*$ in the projective space $\mathbb{P}(V^*) \cong \mathbb{C}P^{\ell-1}$. The dual point configuration $\mathcal{A}^*$ can be used to depict the combinatorial structure of an arrangement in case $\operatorname{rk}(\mathcal{A}) \leq 4$ if the defining forms α_i have real coefficients. (In this case $\mathcal{A}$ is called a *complexified* arrangement.) One merely plots the points α_i in a suitably chosen affine chart $\mathbb{R}^{\ell-1}$ in the real projective space $\mathbb{R}P^{\ell-1}$, for instance by scaling the α_i so that the coefficient of x_1 in each is equal to 1, and then ignoring this coefficient. Dependent flats of rank two (or three) are seen in these affine configurations as lines (or planes) containing more than two (or three) points. These lines and planes are usually explicitly indicated in the picture. This is especially useful for arrangements of rank four. Since the hyperplanes are indicated by points in $\mathbb{R}^3$, they don't obscure the internal structure as a collection of affine planes in $\mathbb{R}^3$ would (see Figure 5). These depictions of projective point configurations are generalized to give *affine diagrams* of arbitrary matroids. Dependent flats are again explicitly indicated with "lines" or "planes," which in the general case may not be straight or flat in the euclidean sense. It is common to refer to flats of rank one,

two, or three in an arbitrary matroid as points, lines, or planes respectively. These diagrams are useful for the study of arrangements which are not complexified real arrangements (see Figures 1 and 2).

1.2. **Basic topological results.** The seminal result in the homotopy theory of arrangements is the calculation of the cohomology algebra of the complement $M = M(\mathcal{A}) := \mathbb{C}^\ell - \bigcup_{i=1}^n H_i$ by Orlik and Solomon [63]. Motivated by work of Arnol'd [1], and using tools established by Brieskorn [10], they gave a presentation of $H^*(M)$ in terms of generators and relations. The presentation $A(\mathcal{A})$ depends only on the underlying matroid $G = G(\mathcal{A})$, and is now called the *Orlik-Solomon (or OS) algebra* of G. Henceforth we will refer to the OS algebra $A(\mathcal{A})$ rather than the cohomology ring $H^*(M)$. The algebra $A(\mathcal{A})$ is defined as the quotient of the exterior algebra on generators $e_1, \ldots, e_n$ by the ideal I generated by "boundaries" of dependent sets of G. See [65] for a precise definition.

This result of [63] gave rise to a collection of "homotopy type" conjectures, which assert that various homotopy invariants of the complement depend only on $G(\mathcal{A})$. A great deal of research in the homotopy theory of arrangements has been focused on conjectures of this type. Note that such conjectures may have "weak" or "strong" solutions: one may show that the invariant depends only on the matroid, or one may give an algorithm to compute the invariant from matroidal data.

The major positive result in this direction is the lattice-isotopy theorem, proved by the second author in [76]. It asserts that the homotopy type, indeed the diffeomorphism type of the complement remains constant through a "lattice-isotopy," that is, a one-parameter family of arrangements in which the intersection lattice, or equivalently, the underlying matroid remains constant.

This result is often recast in terms of matroid realization spaces, which are related to the well-known "matroid stratification" of the Grassmannian. We describe this connection. The defining forms α_i of $\mathcal{A}$ can be identified with row vectors, and thus the arrangement $\mathcal{A}$ can be identified with an $n \times \ell$ matrix R over $\mathbb{C}$. This matrix is called a *realization* of the underlying matroid. Two realizations R and R' are equivalent if there is a nonsingular diagonal $n \times n$ matrix S and a nonsingular $\ell \times \ell$ matrix T such that $R' = SRT$. The corresponding arrangements will then be linearly isomorphic. The set of equivalence classes of realizations of a fixed matroid G is called the (projective) realization space $\mathcal{R}(G)$ of G. Now assume the matrix R has rank ℓ, i.e., that $\mathcal{A}$ is an essential arrangement. Then the column space of R is an ℓ-plane P_R (sometimes denoted $P_{\mathcal{A}}$) in $\mathbb{C}^n$. Note that an isomorphic copy of the arrangement $\mathcal{A}$ inside P_R is formed by the intersection of P_R with the coordinate hyperplanes in $\mathbb{C}^n$. Postmultiplying A by a nonsingular matrix doesn't affect P_R. Thus we see that the realization space $\mathcal{R}(G)$ can be identified with a subset $\Gamma(G)$ of the space of orbits of the diagonal $(\mathbb{C}^*)^n$ action on the Grassmanian $\mathcal{G}_\ell(\mathbb{C}^n)$ of ℓ-planes in $\mathbb{C}^n$. The subsets $\widehat{\Gamma}(G) = \{P_R \mid R \text{ is a realization of } G\} \subseteq \mathcal{G}_\ell(\mathbb{C}^n)$ are called *matroid strata*, although they do not comprise a stratification in the usual sense, since the closure of a stratum may not be a union of strata [85]. These strata play a central role in the theory of generalized hypergeometric functions, especially when the original arrangement $\mathcal{A}$ is generic. The topology of the strata themselves can be as complicated as arbitrary affine varieties over $\mathbb{Q}$ even for matroids of rank three, by a celebrated theorem of Mnëv [59]. These strata are connected by "deletion maps," whose fibers are themselves complements of arrangements [5, 30].

Realizations in $\Gamma(G)$ correspond to arrangements which have the same underlying matroid G, as determined by the arbitrary ordering of the hyperplanes. Thus, for the study of homotopy type as a function of intrinsic combinatorial structure (i.e., without regard to labelling), the true "moduli space" for arrangements should be the quotient of $\mathcal{G}_\ell(\mathbb{C}^n)$ by the action of the $S_n \times (\mathbb{C}^*)^n$. Then linear isomorphism classes of arrangements with isomorphic underlying matroids (or isomorphic intersection lattices) correspond to points of the orbit space $\Gamma(G)/\operatorname{Aut}(G)$.

Randell's lattice-isotopy theorem can be reformulated as follows: two arrangements which are connected by a path in $\widehat{\Gamma}(G)$ (or $\Gamma(G)$) have diffeomorphic complements. Thus one is led to the difficult problem of understanding the set of path components of $\Gamma(G)/\operatorname{Aut}(G)$.

More detailed combinatorial data will suffice to uniquely determine the homotopy type of the complement. For instance, in the case of complexified real arrangements, the defining forms $\alpha_i, 1 \leq i \leq n$ determine an underlying *oriented matroid*. This is most easily described in terms of bases: the matroid $G(\mathcal{A})$ is determined by the collection $\mathcal{B}$ of maximal independent subsets $B \subseteq [n]$. These can naturally be identified with ordered subsets of $[n]$. The *oriented* matroid $\widehat{G}(\mathcal{A})$ is then a partition $\mathcal{B} = \mathcal{B}_+ \cup \mathcal{B}_-$ of the set of ordered bases of $G(\mathcal{A})$ into positive and negative bases, corresponding to the sign of the (nonzero) determinant of the corresponding ordered sets of linear forms. The work of Salvetti [81], as refined by Gelfand and Rybnikov [39], shows that the underlying oriented matroid of a complexified real arrangement uniquely determines the homotopy type of the complement. In fact one can construct a partially ordered set $\mathcal{K}(\widehat{G})$ directly from the oriented matroid $\widehat{G}$ whose "nerve", or collection of linearly ordered subsets, forms a simplical complex homotopy equivalent to the complement. In subsequent work, Björner and Ziegler [7] (see also Orlik [61]) generalized the construction to arbitrary arrangements (or arrangements of subspaces), in terms of combinatorial structures called *2-matroids* [7] or *complex oriented matroids* [93]. They showed that this detailed combinatorial data determines the complement up to piecewise-linear homeomorphism.

The relation between Randell's lattice-isotopy theorem and the combinatorial complexes of [81, 39, 7, 61] has not been fully explored. In particular, it would be interesting to cast the notion of lattice-isotopy in combinatorial terms, i.e., as a sequence of elementary "isotopy moves" on the posets $\mathcal{K}(\widehat{G})$ which leave the homotopy type of the nerve unchanged. A first step in this direction was accomplished in [29]. We pose this as our first open problem.

Problem 1.1. *Prove a combinatorial lattice-isotopy theorem, that "isotopic" (complex) oriented matroids (with the same underlying matroid) determine homotopy equivalent cell complexes.*

1.3. **Homotopy classification.** The fundamental question whether the homotopy type of $M(\mathcal{A})$ is uniquely determined by $G(\mathcal{A})$ was answered in the negative by Rybnikov in [80]. The basic building block of his construction is the MacLane matroid, whose affine diagram is pictured in Figure 1.

For this matroid G, the realization space $\mathcal{R}(G)$ consists of two conjugate complex realizations R and $\overline{R}$, corresponding to arrangements $\mathcal{A}$ and $\overline{\mathcal{A}}$. One can "amalgamate" these realizations along one of the three-point lines (rank-two flats) to form

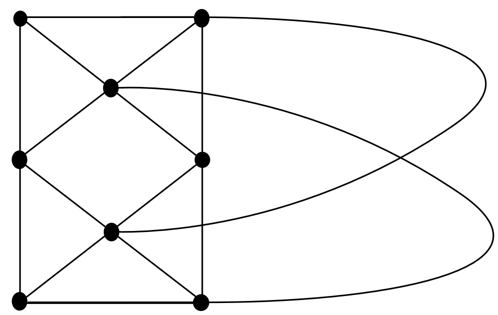


FIGURE 1. The MacLane matroid

arrangements $\mathcal{A} * \mathcal{A}$ and $\mathcal{A} * \overline{\mathcal{A}}$ of rank four with thirteen hyperplanes. These arrangements have the same underlying matroid, of rank four on 13 points, pictured in Figure 2.

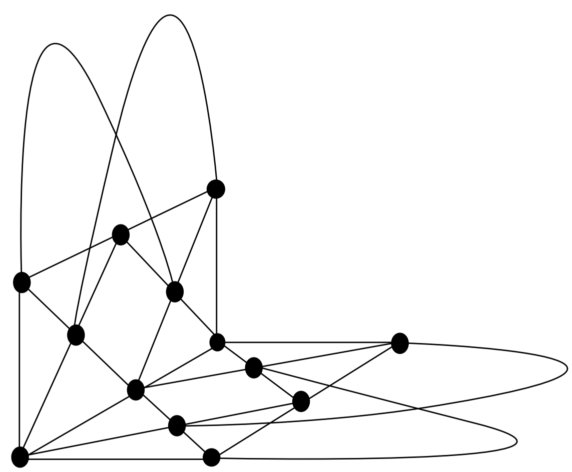


FIGURE 2. The Rybnikov matroid

Rybnikov establishes some special properties of this matroid, for instance, that any automorphism of the OS algebra arises from a matroid automorphism, which must preserve or interchange the factors of the amalgamation. Using these he is able to show that the arrangements $\mathcal{A} * \mathcal{A}$ and $\mathcal{A} * \overline{\mathcal{A}}$ have nonisomorphic fundamental groups, since the first has an automorphism which switches the factors preserving orientations of the natural generators, while the only automorphism of the second which switches factors must reverse orientations. Refer to Section 4.1 for a more detailed description of the fundamental group. Rybnikov actually uses the rank-three truncation of this matroid, and 3-dimensional generic sections of these arrangements, but this operation does not affect the fundamental group.

The last part of Rybnikov's argument is quite delicate and very specialized. None of the known invariants of fundamental groups, for instance those described elsewhere in this paper, will distinguish these two groups.

Problem 1.2. *Find a general invariant of arrangement groups that distinguishes the two Rybnikov arrangements, and generalize his construction.*

To date this is the only known example of this phenomenon. In particular it is not known if this behavior is exhibited by complexified arrangements.

Problem 1.3. *Prove that the underlying matroid of a complexified arrangement determines the homotopy type, or find a counter-example.*

Partial results along these lines were obtained by Jiang and Yau [44] and Cordovil [18]. In [44] a condition on the underlying matroid G is given which implies that the realization space of G is path-connected, so that any two arrangements realizing G have diffeomorphic complements by the lattice-isotopy theorem. In [18] it is shown that complexified arrangements whose underlying matroids are isomorphic via a correspondence which preserves a (geometrically defined) "shelling order" will have identical braid-monodromy groups.

The extent to which arrangements with non-isomorphic matroids can have homotopy equivalent complements has also been studied (see, e.g., [28, 29, 13, 24, 32]) with some degree of success. One approach to this problem is purely combinatorial, namely to classify OS algebras up to graded algebra isomorphism. This approach is adopted in [28, 32, 24]. A powerful invariant is developed in [32], sufficient to distinguish all known non-trivial examples which are not already known to be isomorphic.

At this point all known examples of matroids with isomorphic OS algebras can be explained by two simple operations [35, 72]. The first of these is a construction involving a well-known equivalence of affine arrangements arising from the "cone-decone" construction [65, Prop. 5.1], along with the trivial fact that the complement of the direct sum of affine arrangements, denoted $\coprod$ in [65], is diffeomorphic to the cartesian product of the complements of the factors. In fact this construction can be applied to arbitrary pairs of matroids to yield central arrangements with non-isomorphic matroids and diffeomorphic complements [24, 35]. This construction always yields arrangements with non-connected (i.e., nontrivial direct sum) matroids. Jiang and Yau [45] show that this phenomenon cannot occur in rank three, that is, the diffeomorphism type of the complement of a rank-three arrangement uniquely determines the underlying matroid. Thus the rank-three examples of [29], which have non-isomorphic underlying matroids, have complements which are homotopy equivalent but not diffeomorphic.

The second operation which yields isomorphic OS algebras is truncation. It is shown in [72] that the truncations of two matroids with isomorphic OS algebras will have the same property. (It is not known if truncation preserves homotopy equivalence). These two "moves" suffice to explain the examples produced in [65, 29], indeed all known examples of this phenomenon. Thus it seems an orderly classification of OS algebras may be within reach.

Problem 1.4. *Classify OS algebras up to graded isomorphism.*

In the alternative, we suggest the following.

Problem 1.5. *Find a pair of arrangements with homotopy equivalent complements and whose underlying matroids are non-isomorphic, connected, and inerectible (i.e., not truncations).*

Cohen and Suciu in [12, 13, 14] approach this same problem of homotopy classification using invariants of the fundamental group. Their approach has the advantage that it may also be used to distinguish the complements of arrangements with the same underlying matroid. Some of this work is described elsewhere in this paper. Here we merely remark on the surprising connection described in [14, 55, 50] between the characteristic varieties of [53] arising from the Alexander invariant of the fundamental group, and the resonant varieties of [32], which arise from the OS algebra.

2. Algebraic properties of the group of an arrangement

The topology of hyperplane complements seems to be to a large extent controlled by the fundamental group. These "arrangement groups" have relatively simple global structure, being pieced together out of free groups in a fairly straightforward way (see Sections 4.1 and 3.3), but have surprisingly delicate fine structure. At the time of the writing of [37] there was a great deal of activity around the study of the lower central series of these groups, and connections with rational homotopy theory and Chen's theory of iterated integrals. In this section we report on progress in these areas in the intervening years.

2.1. **The LCS formula, quadratic algebras, rational $K(\pi,1)$ and parallel arrangements.** Discoveries of Kohno [48] and the authors [36] showed that Witt's formula for the lower central series of finitely generated free Lie algebras (or, equivalently, free groups) generalized to a wide class of hyperplane complements. The so-called LCS formula reads

$$\prod_{n\geq 1}(1-t^n)^{\phi_n} = \sum_{i\geq 0} b_i(-t)^i,$$

relating the ranks ϕ_n of factors in the lower central series of the fundamental group $\pi_1(M)$ to the betti numbers $b_i = \dim(A^i(\mathcal{A}))$ of M. In [36, 43] it is shown that this formula holds for all fiber-type arrangements. These are arrangements whose underlying matroids are supersolvable [87]. This result was ostensibly extended to *rational* $K(\pi,1)$ arrangements in [26, 47]. (See also Section 2.2.) We refer the reader to [26, 65] for a precise definition of rational $K(\pi,1)$ arrangement. Briefly, if $\mathcal{S}$ is the 1-minimal model of M (or, equivalently, of $A(\mathcal{A})$), then $\mathcal{A}$ is rational $K(\pi,1)$ if $H^*(\mathcal{S}) \cong A(\mathcal{A})$. It is shown in [26] that fiber-type arrangements are rational $K(\pi,1)$.

The technical results of [36] were used in [38] to show that fundamental groups of fiber-type arrangements (in particular, the pure braid group) are residually nilpotent. This result turned out to be important for the theory of knot invariants of finite type [84].

The situation surrounding the LCS formula was very much in flux during the preparation of [37], a fact reflected in the equivocal footnotes in the table of implications in that paper. The situation has been clarified somewhat in the meantime. Our purpose here is to briefly summarize the current understanding of these issues.

Recall that an arrangement of rank three is *parallel* if for any four hyperplanes of $\mathcal{A}$ in general position, there is a fifth hyperplane in $\mathcal{A}$ containing two of the six pairwise intersections. The OS algebra $A(\mathcal{A})$ is *quadratic* if the relation ideal I (defined in Section 1.2) is generated by its elements of degree two. We will sometimes say $\mathcal{A}$ is quadratic. This is a combinatorial condition, which will be

discussed in further detail in Section 3.2. In general the quotient of the exterior algebra $\Lambda(e_1,\dots,e_n)$ by the ideal generated by the degree two elements of I is called the *quadratic closure* of $A(\mathcal{A})$, denoted $\overline{A}(\mathcal{A})$. Here is a summary of cogent results established in [26, 27].

(i) If $\mathcal{A}$ is a rational $K(\pi,1)$ arrangement, then $\mathcal{A}$ is quadratic.
(ii) Every parallel arrangement is quadratic.
(iii) Every rational $K(\pi,1)$ arrangement satisfies the LCS formula.
(iv) Every quadratic arrangement satisfies the LCS formula at least to third degree.

In [37] we cited an unpublished note which claimed that every parallel arrangement is a rational $K(\pi,1)$. Using the construction of [26], in 1994 Falk wrote a *Mathematica* program to compute ϕ_4, and checked the smallest example of a parallel, non-fiber-type arrangement of rank 3. This arrangement, labelled X_2 in [37], consists of the planes $x\pm z=0, y\pm z=0, x+y\pm 2z=0$, and $z=0$, and is pictured in Figure 3. We obtained the result $\phi_4=15$, whereas the LCS formula would predict $\phi_4=10$.

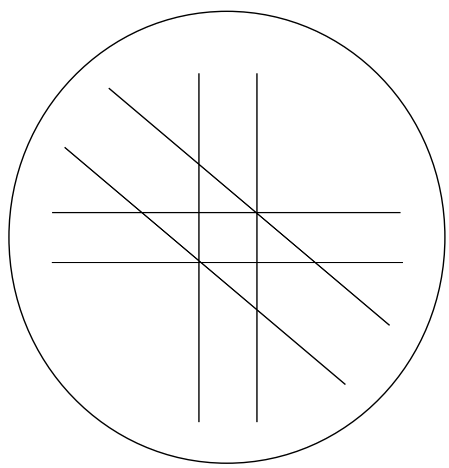


FIGURE 3. The arrangement X_2

So the implications

$$\text{parallel} \implies \text{rational } K(\pi,1),$$
$$\text{quadratic} \implies \text{rational } K(\pi,1),$$
$$\text{parallel} \implies \text{LCS},$$

and

$$\text{quadratic} \implies \text{LCS}$$

recorded in [37] are all false.

Subsequently, work of Shelton-Yuzvinsky [82], and Papadima-Yuzvinsky [67] provided further clarification. Let $\mathcal{L}$ denote the holonomy Lie algebra of M, the quotient of the free Lie algebra on generators $x_1,\dots x_n$ by the image of the map $H_1(M)\to\Lambda^2(H_1(M))$ dual to the cup product. Let $U=U(\mathcal{A})$ be its universal

enveloping algebra, a dual object to the 1-minimal model $\mathcal{S}$. The Hilbert series of U is $\prod_{n\geq 1}(1-t^n)^{-\phi_n}$. Kohno constructs a chain complex (R,δ) which, when exact, forms a resolution of $\mathbb{Q}$ as a trivial U-module. In this case $\mathcal{A}$ is a rational $K(\pi,1)$ arrangement, and the LCS formula holds.

Shelton and Yuzvinsky [82] realized that $U(\mathcal{A})$ is the Koszul dual of the quadratic closure of $A(\mathcal{A})$. We refer the reader to [82] for a precise definition; loosely speaking, the defining relations for the Koszul dual U form the orthogonal complement to those of $\overline{A}(\mathcal{A})$ inside the tensor product $T_2(A^1(\mathcal{A}))$. They observed that the Aomoto-Kohno complex (R,δ) is the usual Koszul complex of U, and thus is exact if and only if U is a Koszul algebra — U is *Koszul* iff $\mathrm{Ext}_U^{p,q}(\mathbb{Q},\mathbb{Q})=0$ unless $p=q$. It follows from this that $A(\mathcal{A})$ is a quadratic algebra. (This observation was also made by Hain [41].) The LCS formula is then a consequence of Koszul duality. They give a combinatorial proof that $A(\mathcal{A})$ is quadratic and that $U(\mathcal{A})$ is Koszul if $\mathcal{A}$ is a supersolvable arrangement.

The results of [82] were strengthened and extended in [67] to give a description of $H^*(\mathcal{S})$ in terms of Koszul algebra theory, for more general spaces. In particular, it is shown in [67] that $\mathcal{A}$ is rational $K(\pi,1)$ if only if the OS algebra is Koszul. In addition, Papadima and Yuzvinsky gave an alternate proof that the arrangement X_2 above fails the LCS formula. Finally, using a "central-to-affine" reduction argument, they were able to prove the following.

Theorem 2.1. [67] *For arrangements of rank three, the LCS formula holds if and only if the arrangement is fiber-type.*

Peeva [71] applies techniques of commutative algebra and Gröbner basis theory to obtain a short proof that supersolvable arrangements satisfy the LCS formula, in addition to other related computational results.

In research closely related to the lower central series of arrangement groups, Kohno used the iterated integral/holonomy Lie algebra approach to construct representations of the (pure) braid group, and more generally to study the monodromy of local systems over hyperplane complements. This work is also closely tied to the theory of generalized hypergeometric functions. See [49] for a description of these developments. Cohen and Suciu pursued similar ideas using methods more closely connected to those of [36] in [15].

2.2. **The D_n reflection arrangements.** The fundamental groups of the reflection arrangements of type D_n have been studied using some of the technical machinery of [36]. Note that these arrangements, for $n>3$, are not supersolvable. The author of [58] constructs a presentation which he claims presents these fundamental groups as "almost direct products" in the sense of [36, 15]. He used this to show that these groups are residually nilpotent. In 1994 we tried to use this presentation to get more precise calculations for the lower central series of these groups, at least for $n=4$. In fact we found that the presentation in [58] is not correct. Even for the D_3 arrangement, which is supersolvable, the results one deduces from [58] do not jibe with the LCS formula, which is known to hold for D_3. In [56] Liebman and Markushevich adopt a different approach and derive a different presentation to show that the D_n arrangement groups are residually nilpotent.

It was in the course of this research that we started computing ϕ_4 by machine. In addition to finding the counterexample X_2 described above, we also computed $\phi_4=183$ for the D_4 reflection arrangement. The LCS formula yields $\phi_4=186$.

So the D_4 arrangement fails the LCS formula, contrary to another assertion [46] reported on in [37].

The work of Shelton and Yuzvinsky [82] make it clear why the argument of [46] for the LCS formula for the D_n reflection arrangements fails: these arrangements, for $n > 3$, do not have quadratic OS algebras, by [26]. Hence the Aomoto-Kohno complex $R.$ cannot be exact for these arrangements.

So we are left with no examples of arrangements which are not supersolvable, yet are rational $K(\pi,1)$, and no examples of arrangements satisfying the LCS formula which are not rational $K(\pi,1)$.

Problem 2.2. *Find examples of non-supersolvable or non-rational $K(\pi,1)$ arrangements satisfying the LCS formula, or prove that such examples do not exist.*

2.3. **Work of Cohen and Suciu on the Chen groups.** As noted above, the ranks of the quotients in the lower central series of fiber-type arrangements are determined by the betti numbers of the complement. From this point of view, the pure braid groups look like products of free groups (though they are not; see [38].) In the last few years, Cohen and Suciu have introduced the Chen groups into the study of arrangements, providing a computable tool for distinguishing similar arrangements.

The Chen groups of a group G are the lower central series quotients of G modulo its second commutator subgroup G''. If for any group G we let $\Gamma_k(G)$ denote the k^{th} lower central series subgroup, then the homomorphism $G \to G/G''$ induces an epimorphism

$$\frac{\Gamma_k(G)}{\Gamma_{k+1}(G)} \to \frac{\Gamma_k(G/G'')}{\Gamma_{k+1}(G/G'')} = k^{th} \text{ Chen group}$$

Thus the ranks ϕ_k of quotients of lower central series groups are no less than the corresponding ranks θ_k of Chen groups. In the case of the pure braid group, the ranks θ_k are determined in [12]; they are given by the generating function

$$\sum_{k=2}^{\infty} \theta_k t^{k-2} = \binom{n+1}{4} \cdot \frac{1}{(1-t)^2} - \binom{n}{4}$$

In particular, these numbers differ from those for the product of free groups, providing a tidy proof that the pure braid groups are not such products.

Cohen and Suciu [11] provide a detailed study of these groups including a method for their computation from a presentation of the Alexander invariant (see the discussion of presentations of the fundamental group below.) It is interesting that while these groups are very effective in distinguishing similar groups, there is not yet an example of combinatorially equivalent arrangements with different Chen ranks. In particular, they do not distinguish the examples of Rybnikov [80] of combinatorially equivalent, homotopically different arrangements (see Section 1.3).

2.4. **Cohomological properties of the fundamental group.** In 1972 Deligne [21] proved that for a complexification of a real simplicial arrangement, the complement M is aspherical (also expressed by saying that M is a $K(\pi,1)$ space.) That is, the universal cover of M is contractible. Since all real reflection arrangements are simplicial, this solved a question raised and partially answered by Brieskorn in [9]. The original study of this sort of problem was the work of Fadell and Neuwirth [25] on the pure braid group. Following [86], the authors introduced in [36] the notion of fiber-type arrangement and observed that for this class M is aspherical, essentially

by the iterated fibration argument of Fadell and Neuwirth. So it is natural to ask: for what arrangements is M aspherical? It is known by work of Hattori [42] that not all are — the arrangement defined by $Q = xyz(x+y+z)$ is the simplest example.

Here we wish to touch upon the algebraic consequences of asphericity. Now if M is aspherical, the (known) cohomology of M is isomorphic to the cohomology of the group. Since M has cohomological dimension $\mathrm{rk}(\mathcal{A}) < \infty, \pi_1(M)$ does also. In addition, $\pi_1(M)$ has no torsion, and there is a $K(\pi, 1)$ space, $\pi = \pi_1(M)$, with the homotopy type of a finite complex (namely, M). So here is another open problem:

Problem 2.3. *Are all arrangement groups torsion-free?*

The answer is of course yes for real reflection arrangements and for fiber-type (or supersolvable) arrangements. One approach to this question is to show that all arrangement groups are orderable. Here we say a group G is *orderable* provided that there is a linear order $<$ on G so that $g < h$ implies $cg < ch$ for all $c \in G$. It follows easily that an orderable group has no torsion. The braid group was shown orderable by Dehornoy in [20]; at the Tokyo meeting L. Paris proved that the group of a fiber-type arrangement is orderable [68]. It is not known whether all arrangement groups are orderable. Note that the group of an arrangement has a finite presentation of a fairly restricted type, as described in Section 4.1, and that the relators all lie in the commutator subgroup.

There are some useful observations concerning these ideas in [77]. For instance, we have the following theorem.

Theorem 2.4. *For $j \geq 2$ the Hurewicz map*

$$\phi : \pi_j(M) \to H_j(M)$$

is trivial.

As a consequence, the second homology of $\pi_1(M)$ is isomorphic to $H_2(M)$. In addition, it is mentioned there that the arrangement defined by

$$Q = xyz(y+z)(x-z)(2x+y)$$

has the property that there is no arrangement with aspherical complement with the same intersection lattice in rank one and two. The following result is also proved in [77].

Theorem 2.5. *The complement of a central arrangement of rank three is aspherical provided that the fundamental group has cohomological dimension three and is of type FL.*

A group π is *type FL* provided that $\mathbb{Z}$ (as a trivial $\mathbb{Z}[\pi]$-module) has a finite resolution by free $\mathbb{Z}[\pi]$-modules. An equivalent statement is that there should exist a *finite* CW complex which is a $K(\pi, 1)$-space. Theorem 2.5 shows that for central rank three arrangements asphericity is determined by the fundamental group.

3. Arrangements with aspherical complements

Much of the early history of the topology of arrangements revolves around the "$K(\pi, 1)$ problem," the problem of determining which arrangements have aspherical complements. (Such an arrangement is called a $K(\pi, 1)$ *arrangement.*) This history is described in some detail in [37] (see also Section 2.4). In addition, we

proved an *ad hoc* necessary condition [37, Thm. 3.1] for asphericity involving "simple triangles," and introduced the notion of formal arrangement, which was shown to be a necessary condition for $K(\pi, 1)$ and rational $K(\pi, 1)$ arrangements. A great deal of progress was made in these areas in the intervening years, which we report on in this section.

3.1. **Free arrangements are not aspherical.** In our earlier survey, we highlighted the Saito conjecture, that all free arrangements are aspherical. In 1995 Edelman and Reiner [23] provided counterexamples, which we briefly describe.

Let S denote the polynomial ring of V. A linear map $\theta : S \to S$ is a derivation if for $f, g \in S$, we have $\theta(fg) = f\theta(g) + g\theta(f)$. The module of $\mathcal{A}-$derivations is defined by

$$D(\mathcal{A}) = \{\theta \mid \theta(Q) \in QS\}$$

where Q is the defining polynomial of the arrangement. Then the arrangement is *free* provided that $D(\mathcal{A})$ is a free S-module.

It is known [86] that reflection arrangements are free; for their many pleasant properties see [65]. In 1975 K. Saito conjectured that free arrangements should be aspherical. In their study of tilings of centrally symmetric octagons in [23], Edelman and Reiner found the family of arrangements given by

$$Q(\mathcal{A}_\alpha) = xyz(x-y)(x-z)(y-z)(x-\alpha y)(x-\alpha z)(y-\alpha z)$$

with $\alpha \in \mathbb{R}$. They proved that the corresponding arrangements are free for all α, while they are not aspherical for $\alpha \neq -1, 0, 1$. The proof of freeness is direct, using addition-deletion [65, Theorem 4.51] while the non-asphericity follows from the "simple triangle" criterion of [37]. The counter-example $\mathcal{A}_{-2}$ is pictured in Figure 4.

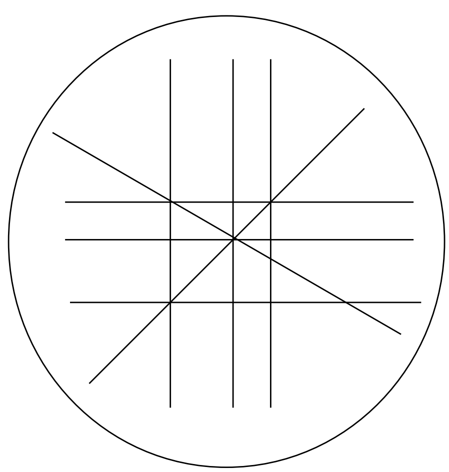


FIGURE 4. Free but not $K(\pi, 1)$

3.2. **Formality and related concepts.** The fundamental group of arrangement is determined by a generic 3-dimensional section. Based on the idea that $K(\pi, 1)$ arrangements should be extremal in some sense, we developed the notion of formal arrangement in [37]. This has been the subject of several papers since [5, 8, 91, 33], which provide a better understanding of the concept. Here is a "modern" definition, equivalent to the original from [37].

Let $\Phi : \mathbb{C}^n \to V^*$ be given by $\Phi(x) = \sum_{i=1}^n x_i \alpha_i$, where the α_i are the defining forms for $\mathcal{A}$. Let $K = \ker(\Phi)$ and let F be the subspace of K spanned by its elements of weight three (i.e., having three nonzero entries). Then the arrangement $\mathcal{A}$ is *formal* if $F = K$.

The orthogonal complement $K^\perp \subseteq \mathbb{C}^n$ coincides with the point $P_{\mathcal{A}} \in \mathcal{G}_\ell(\mathbb{C}^n)$ defined in Section 1.2. Thus the arrangement $\mathcal{A}$ is isomorphic to the arrangement in $K^\perp$ formed by the coordinate hyperplanes. In the same way, the orthogonal complement $F^\perp \supseteq K^\perp$ defines an arrangement $\mathcal{A}_F$, called the *formalization* of $\mathcal{A}$. So $\mathcal{A}$ is formal if and only if $\mathcal{A} = \mathcal{A}_F$. If $\mathcal{A}$ is not formal, $\mathcal{A}_F$ has strictly greater rank, and $\mathcal{A}$ is a (not necessarily generic) section of $\mathcal{A}_F$. Also, $\mathcal{A}$ and $\mathcal{A}_F$ have isomorphic generic "planar" (i.e., rank-three) sections.

These properties of formalization were asserted in [37], but the arguments we had in mind were not correct. The clarification described here is due to Yuzvinsky [91]. Examples in [74] show that non-formal arrangements need not be generic sections of their formalizations. The arrangement of Example 2.19 of [74] has the property that the *free* erection of the underlying matroid is not realizable, but (contrary to the assertion in [74]) there is nevertheless a realizable (formal) erection. Matroid "erection" is the reverse of (corank one) truncation; truncation is the matroid-theoretic analogue of generic section. The free erection of an erectible matroid is the unique erection with "the most general position" — see [89].

These observations are enough to establish the following results from [37]. The third assertion follows immediately from the second.

(i) If $\mathcal{A}$ is a $K(\pi, 1)$ arrangement, then $\mathcal{A}$ is formal.
(ii) If $\mathcal{A}$ is quadratic, then $\mathcal{A}$ is formal.
(iii) If $\mathcal{A}$ is a rational $K(\pi, 1)$ arrangement, then $\mathcal{A}$ is formal.

We asked whether free arrangements are also necessarily formal. This was established by Yuzvinsky.

Theorem 3.1. [91] *If $\mathcal{A}$ is a free arrangement, then $\mathcal{A}$ is formal.*

The preceding result was generalized by Brandt and Terao [8]. They define the notion of k-formal arrangement. A formal arrangement has the property that all relations among the defining equations are consequences of relations which are "localized" at rank-two flats, in the sense that an element of K of weight three gives rise to a three-element subset of a rank-two flat. A formal arrangement is 3-*formal* if all relations among these local generators of $F = K$ are themselves consequences of relations which are localized at rank-three flats of $\mathcal{A}$. This construction is iterated to define the notion of k-formal arrangement for every $k \geq 2$. See [8] for the precise definition. An arrangement of rank r is automatically k-formal for every $k \geq r$. The original notion of formality coincides with the case $k = 2$.

Theorem 3.2. [8] *If $\mathcal{A}$ is a free arrangement of rank r, then $\mathcal{A}$ is k-formal for every $2 \leq k < r$.*

The converse is false [8].

Related work appears in [5], where the authors show that the discriminantal arrangements of Manin and Schechtman [57] (see Section 3.4.2) are formal, and the "very generic" discriminantal arrangements are 3-formal, though none are free.

An arrangement is *locally formal* [91] if, for every flat $X \subseteq [n]$, the arrangement $\mathcal{A}_X = \{H_i \mid i \in X\}$ is formal. Since freeness, quadraticity, and $K(\pi, 1)$-ness are all "hereditary properties," in that they are inherited by the localizations $\mathcal{A}_X$, one has that every free, quadratic, or $K(\pi, 1)$ arrangement is locally formal.

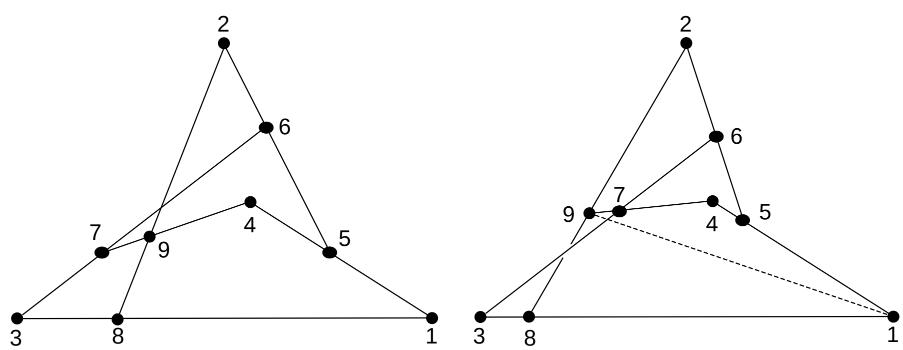
We asked in [37] whether formality is a "combinatorial property", depending only on the underlying matroid. Yuzvinsky constructed counter-examples in [91].

Theorem 3.3. [91] *There exist arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ with the same underlying matroid, such that $\mathcal{A}_1$ is formal and $\mathcal{A}_2$ is not formal.*

In Figure 5 are the dual point configurations of Yuzvinsky's arrangements. The dotted line in Figure 5(b) indicates where to "fold" the configuration to erect it to a rank-four configuration. The nontrivial planes in the erection are

12389, 12456, 13458, 13678, 14579, 23567, 24789, 25689, and 34679.

Note that these two configurations are lattice-isotopic (over $\mathbb{C}$), so neither is free or $K(\pi, 1)$.



(a) A formal configuration (b) A non-formal configuration

Figure 5. Formality is not matroidal

If $\mathcal{A}$ is not formal, then the underlying matroid of $\mathcal{A}$ is a *strong map image* (under the identity map) of that of $\mathcal{A}_F$ (see [66] for the general definition), and the two matroids have the same rank-three truncations. These combinatorial properties gave rise to several attempts to replace the notion of formality with some clearly matroidal condition, and strengthen Theorem 3.1 and assertion (i) above. For example one can ask for conditions on a matroid G so that every (complex) realization of G is formal. One is naturally led to the notion of line-closure.

Let G be a matroid on ground set $[n]$. The *line-closure* of a subset S of $[n]$ is the smallest subset of $[n]$ which contains every line (that is, rank-two flat) spanned by points of S. A set is line-closed if it is equal to its line-closure. The matroid G

is *line-closed* if every line-closed subset of $[n]$ is a flat of G. In his current work in progress [33], the first author has established the following result.

Theorem 3.4. *An arrangement $\mathcal{A}$ is quadratic only if the underlying matroid $G(\mathcal{A})$ is line-closed.*

Corollary 3.5. *The underlying matroid of a rational $K(\pi,1)$ arrangement is necessarily line-closed.*

The converse of Theorem 3.4, that $\mathcal{A}$ is quadratic when $G(\mathcal{A})$ is line-closed, is very likely also true. A crucial step in the proof is yet to be completed, however, so this assertion remains an open problem.

Yuzvinsky [90] defined a *formal matroid* to be a matroid G possessing a basis (of $\mathrm{rk}(G)$ points) whose line-closure is $[n]$. Every line-closed matroid is formal in this sense. In fact a matroid G is line-closed if and only if the line-closure of *every* basis of each flat X is equal to X. Every realization of a formal matroid is formal.

In [33] we define a matroid G to be *taut* if G is not a strong map image of a matroid G' of greater rank with the same points and lines, and *locally taut* if every flat of G is taut. Every line-closed matroid is locally taut, in fact every formal matroid is taut. Every realization of a (locally) taut matroid is (locally) formal. There exist matroids which are taut but not formal [19]. A weak version of the first part of the following problem was suggested by Yuzvinsky in his talk [90].

Problem 3.6. *Prove that the matroid of a free or $K(\pi,1)$ arrangement is necessarily taut.*

Joseph Kung has pointed out to us that a locally taut matroid is uniquely determined by its points and lines, which suggests the following interesting problem.

Problem 3.7. *Prove that the underlying matroid of a locally formal arrangement (e.g. a free or $K(\pi,1)$ arrangement) is uniquely determined by its points and lines.*

This last problem is a variant on the following questions from [37], the first of which is Terao's Conjecture, and both of which remain open.

Problem 3.8. *Prove that freeness and $K(\pi,1)$-ness of arrangements are matroidal properties.*

We will refrain from discussing Terao's Conjecture further, except to pose a weak version which fits the spirit of this paper, and is interesting in its own right.

Problem 3.9. *Prove that freeness is preserved under lattice-isotopy.*

3.3. **Tests for asphericity.** Some progress was also made on the problem of finding sufficient conditions for an arrangement to be $K(\pi,1)$. The main results are the weight test of [31] and its application to factored arrangements by Paris [69]. A new technique involving modular flats was recently discovered and presented at the conference [70, 35].

The complement M of a 2-dimensional affine arrangement $\mathcal{A}$ is built up out of $K(\pi,1)$ spaces, specifically (r,r) torus link complements, in a relatively simple way, as is reflected in the Randell-Salvetti-Arvola presentations (see Section 4.1). In fact this structure mirrors precisely constructions from geometric group theory related to complexes of groups. This observation allows one to construct a relatively well-behaved cell complex which has the homotopy type of the universal cover of M, and to apply the weight test of Gersten and Stallings [83] to derive a test for asphericity of M.

Theorem 3.10. [31] *If $\mathcal{A}$ is a complexified affine arrangement in $\mathbb{C}^2$ that admits an $\mathcal{A}$-admissible, aspherical system of weights, then $\mathcal{A}$ is a $K(\pi, 1)$ arrangement.*

The question remains what an $\mathcal{A}$-admissible, aspherical system of weights is. This involves the complex B of bounded faces in the subdivision of $\mathbb{R}^2$ determined by $\mathcal{A}$. A weight system is an assignment of a real number weight to each "corner" of each 2-cell in B. The system is aspherical if the sum of weights around any d-gon at most $d-2$. The system is $\mathcal{A}$-admissible if certain sums of weights at vertices of Γ are at least 2π. See [31] for more detail.

The universal cover complex constructed in [31] may be used in some cases to construct explicit essential spheres showing that M is not aspherical. Radloff [74] used this method to prove some necessary conditions for $K(\pi, 1)$-ness, along the lines of the "simple triangle" test of [37], and found several new examples of non-$K(\pi, 1)$ arrangements.

Falk and Jambu introduced the notion of *factored* arrangement in [34], originally in an attempt to find a combinatorial criterion for freeness. A *factorization* of an arrangement $\mathcal{A}$ is a partition of $[n]$ such that each flat of $G(\mathcal{A})$ of rank p meets precisely p blocks, and meets one of them in a singleton, for each p. This property is necessary and sufficient for the OS algebra $A(\mathcal{A})$ to have a complete tensor product factorization - see [6, 34, 88, 65]. When $\mathcal{A}$ has a factorization, we say $\mathcal{A}$ is *factored*.

Paris realized that a factorization of a rank-three arrangement provides a template for a very simple $\mathcal{A}$-admissible, aspherical weight system.

Theorem 3.11. [69] *If $\mathcal{A}$ is a factored, complexified arrangement in $\mathbb{C}^3$, then $\mathcal{A}$ is a $K(\pi, 1)$ arrangement.*

Every supersolvable arrangement is factored, so this result provides a new, wider class of $K(\pi, 1)$ arrangements, at least in rank three.

Problem 3.12. *Show that factored arrangements of arbitrary rank are $K(\pi, 1)$.*

A flat X of a matroid G is *modular* if $\mathrm{rk}(X \vee Y) + \mathrm{rk}(X \wedge Y) = \mathrm{rk}(X) + \mathrm{rk}(Y)$ for every flat Y. The following result was discovered independently by Paris and Falk-Proudfoot

Theorem 3.13. [70, 73, 35] *If X is a modular flat of arbitrary rank in $G(\mathcal{A})$, then there is a topological fibration $M(\mathcal{A}) \to M(\mathcal{A}_X)$ whose fiber is the complement of a projective arrangement.*

This generalizes the corank-one case, which gives rise to fiber-type arrangements, established in [87]. The new result can be used to construct or recognize $K(\pi, 1)$ arrangements if the base (whose matroid is the modular flat X) and fiber (whose matroid is the complete principal truncation of $G(\mathcal{A})$ along X) are known to be $K(\pi, 1)$. This method is used to construct some interesting new examples in [35]. Refer to Paris' paper [68] in this volume for more details.

3.4. **Some crucial examples.** In this section we want to briefly discuss some specific and interesting types of arrangements for which the $K(\pi, 1)$ problem is unsolved. These might be regarded as test subjects for new techniques; they qualify as "the first unknown cases."

First we cite another improvement to the table of implications in [37]. Recall the definition of parallel arrangement from Section 2.1. In [37] we had listed the implication "parallel $\implies$ $K(\pi, 1)$" as "not known, of significant interest." In

unpublished work, Luis Paris has shown this implication to be false. Specifically, he showed that the Kohno arrangement X_2 (defined in Section 2.1) is not $K(\pi, 1)$. The proof establishes that the fundamental group contains a subgroup isomorphic to $\mathbb{Z}^4$; the result then follows from [37, Thm. 3.2]. The copy of $\mathbb{Z}^4$ is generated by a, b, c, and the commutator $[d, e]$, where a, b, c, d, and e are the canonical generators corresponding to the hyperplanes $x \pm z = 0,\ z = 0$, and $x + y \pm 2z = 0$ respectively.

3.4.1. *Complex reflection arrangements.* Fadell and Neuwirth showed in 1962 that the complement of the A_ℓ reflection (or braid) arrangement is $K(\pi, 1)$. In 1973 Brieskorn proved this for many real reflection arrangements, followed soon thereafter by Deligne's proof of the general case. Orlik and Solomon extensively studied arrangements of hyperplanes invariant under finite groups generated by complex reflections (see [65, Chapter 6]). It is natural to ask if all such arrangements are aspherical. We believe the conjecture that they are is due to Orlik, though it was proposed long before it ever appeared in print. It is known [65] that the answer is affirmative in all cases except six exceptional, non-complexified arrangements, some of which have rank three. The proofs for the known cases use a variety of techniques, and essentially proceed from the Shepard-Todd classification of irreducible unitary reflection groups (see, e.g., [65]). What seems to be missing is a unifying property, similar to the simplicial property for real reflection arrangements exploited by Deligne. The closest approach to this goal is the work reported in [65, p. 265] which proves the asphericity of arrangements associated to Shephard groups (symmetry group of a regular convex polytope.) Here the problem is reduced to the (already solved) problem for an associated real reflection arrangement.

Problem 3.14. *Give a uniform proof that all unitary reflection arrangements are $K(\pi, 1)$.*

3.4.2. *Discriminantal arrangements.* Experience seems to show us that questions involving asphericity are quite complex for all arrangements but tractible for restricted classes (reflection, fiber-type, generic). One interesting class is that of the discriminantal arrangements introduced by Manin and Schechtmann [57]. Rather than give the full definition here we will describe the rank three examples, where the problem is already interesting.

Consider a real affine arrangement of lines in the plane, obtained by taking a collection of n points, no three of which are collinear, and drawing all $\binom{n}{2}$ lines through pairs of these points. Then embed this configuration in the plane $z = 1$ in three-space and cone over the origin to obtain a central real three-arrangement. Then complexify.

This process can result in arrangements with distinct matroidal and topological structure, even for fixed n [30, 5]. The discriminantal arrangements are obtained from "very generic" collections of points, for which no three of the $\binom{n}{2}$ lines are concurrent except at the original n points.

The arrangement $C(4)$ is linearly equivalent to the braid arrangement of rank three. An easy calculation shows that the Poincaré polynomial associated to the cohomology of $C(n)$ does not factor over $\mathbb{Z}$ for $n \geq 5$, so that these arrangements are not free and are not of fiber-type. Also $C(n)$ is not simplicial for $n \geq 5$. The arrangements $C(n)$ for $n \geq 6$ are not aspherical, by [37, Thm. 3.1].

For $n = 5$, one obtains a complexified central three-arrangement of 10 planes. This arrangement is not factored. More generally $C(5)$ does not support an admissible, aspherical system of weights, so the weight test fails. On the other hand, all of the standard necessary conditions for asphericity hold.

Problem 3.15. *Determine whether the discriminantal arrangement* $C(5)$ *is* $K(\pi, 1)$.

A solution to this problem would also determine whether the space of configurations of six points in general position in $\mathbb{C}P^2$ is aspherical [30], a result which would be of significant interest.

3.4.3. *Deformations of reflection arrangements.* A "deformation" of a reflection arrangement is an affine arrangement with defining equations of the form

$$\alpha_i(x_1, \ldots, x_\ell) = c_{ij},$$

where the α_i are the positive roots in some root system, and $c_{ij} \in \mathbb{R}$. This class of arrangements is of great interest to combinatorialists, and is the subject of the paper of Athanasiadis in this volume [4].

As is our custom, we "cone" to obtain a central arrangement. For instance, based on the root system of type B_2, we obtain the B_2 *Shi arrangement*, defined by the polynomial

$$Q = xyz(x+y)(x-y)(x-z)(y-z)(x+y-z)(x-y-z).$$

(Shi arrangements are obtained by setting $c_{i1} = 0$ and $c_{i2} = 1$ for all i.) This nine-line complexified arrangement has a factorization, given by the partition

$$\{\{4\}, \{1, 2, 5, 7\}, \{3, 6, 8, 9\}\},$$

and is therefore a $K(\pi, 1)$ arrangement. On the other hand, the Shi arrangement constructed in a similar way from the root system of type G_2 is not factored or simplicial, and has no simple triangle.

Problem 3.16. *Decide whether the* G_2 *Shi arrangement is* $K(\pi, 1)$.

More generally, we propose the following.

Problem 3.17. *Decide which Shi arrangements are* $K(\pi, 1)$.

4. Topological properties of the group of an arrangement

At the time of the publication of [37], a presentation of the fundamental group of the complement of a complexified arrangement had been derived [75]. In the meantime, a similar presentation was found for arbitrary complex arrangements [3], and several different "spines" for the complement, some of them modelled on group presentations, were constructed [81, 29, 13, 54]. These group presentations have been used to study the Milnor fibration and Alexander invariants of the complement. We report briefly on these ideas here.

4.1. **Presentations of π_1.** We have seen earlier in the discussion of the lower central series, Chen groups and group cohomology that certain classes of arrangements (fiber-type, simplicial) have well-behaved fundamental groups. Due to work of Arvola [3], Randell [75] and Salvetti [81] an explicit presentation of $\pi_1(M)$ can be written. See [65, Section 5.3] for a clear exposition of Arvola's presentation for any complex arrangement, and [29] for the explicit presentation and some applications of Randell's presentation, which holds for complexified arrangements and is

naturally simpler than the general case. A different approach, using the notion of "labyrinth," is adopted by Dung and Vui in [22] to arrive at similar presentations for arbitrary arrangement groups.

In these presentations one first takes a planar section (or, more precisely, the projective image), so that one is working with an affine arrangement in $\mathbb{C}^2$. Then there is one generator for each line of the arrangement, and one set of relations for each intersection. In all cases the relations consist entirely of commutators, but to date this has not shed much light on the questions of group cohomology, torsion in the fundamental group, or other properties (such as orderability) of the fundamental group. A general theme for questions is: to what extent do arrangement groups mimic the properties of the pure braid groups.

The concept of braid monodromy was introduced by B. Moishezon [60]. Libgober showed in [54] that the braid monodromy presentation of the fundamental group yields a two-complex with the homotopy type of the complement of an algebraic curve (e.g., a line arrangement) transverse to the line at infinity.

Motivated in part by [54], the first author showed in [29] that for arbitrary line arrangements the 2-complex modelled on the presentation of [75] serves as an efficient model for constructing the homotopy type of the complement (in the case of 3-arrangements). This construction was then used to construct a number of examples with different intersection lattice but same homotopy type (see also Section 1.3).

In related work Cohen and Suciu [13] have given an explicit description of the braid monodromy of a complex arrangement, using Hansen's theory of polynomial covering maps. They show that the resulting presentation of the fundamental group is equivalent to the Randell-Arvola presentation via Tietze transformations that do not affect the homotopy type of the associated 2-complex. It follows that the complement is homotopy equivalent to the 2-complex modelled on either of these presentations, generalizing the result of [29]. For this work Cohen and Suciu used extensively the concept of *braided wiring diagram*, which we briefly describe below. The notion of braided wiring diagram generalizes Goodman's concept of wiring diagram [40], and was earlier considered for arrangements in [17]. (Wiring diagrams appear in combinatorics as geometric models for rank-three oriented matroids.) The presentations of [75] and [3] use versions of this idea. In brief, the braided wiring diagram can be thought of as a template for the fundamental group (or, for line arrangements, the homotopy type.)

Here is a sketch of the construction. For examples and further details, in particular, a beautiful derivation using polynomial covering space theory, see [13]. Since we are interested in the fundamental group, consider an affine arrangement $\mathcal{A}$ in $\mathbb{C}^2$. Choose coordinates in $\mathbb{C}^2$ so that the projection to the first coordinate is generic. Suppose that the images $y_1, \ldots, y_n$ of the intersections of the lines have distinct real parts. Choose a basepoint $y_0 \in \mathbb{C} \setminus \{y_1, \ldots, y_n\}$, and assume the real parts of y_i are decreasing with i. Let ξ be a smooth path which begins with y_0 and passes in order through the y_i, horizontal near each y_i. Then the braided wiring diagram is $\mathcal{W} = \{(x, z) \in \xi \times \mathbb{C} \mid Q(x, z) = 0\}$. (Recall that Q is the defining polynomial of the arrangement.)

This braided wiring diagram should be viewed as a picture of the braid monodromy of the fundamental group of the arrangement (or as a picture of the fundamental group itself). In a sense, it carries the attaching (or amalgamating) information as one computes the fundamental group using the Seifert-Van Kampen

theorem. Each actual node in the wiring diagram gives a set of relators, as does each crossing. In particular, it is shown in [13] that the braided wiring diagram recovers the Arvola or Randell presentation of $\pi_1(M)$. Indeed, in the real case, the braided wiring diagram can be identified with the usual drawing of the arrangement in $\mathbb{R}^2$.

As is the case with ordinary braids, there are "Markov moves" with which one can modify such a wiring diagram to realize any braid-equivalence of the underlying braid monodromies. These are given explicitly in [13]. Rudimentary moves of this type, called "flips," first appeared in [29]. Among the consequences we note the following results which relate braid monodromy and braided wiring diagrams to lattice isotopy of line arrangements (that is, arrangements in $\mathbb{C}^2$).

Theorem 4.1. [13] *Lattice-isotopic arrangements in $\mathbb{C}^2$ have braided wiring diagrams which are related by a finite sequence of Markov moves and their inverses.*

Theorem 4.2. [13] *Line arrangements with braid-equivalent monodromies have isomorphic underlying matroids.*

4.2. **The Milnor fiber.** The defining polynomial $Q = \prod_{i=1}^{n} \alpha_i$ is homogeneous of degree n and can be considered as a map

$$Q : M \to \mathbb{C}^*$$

It is well-known that this map is the projection of a fiber bundle, called the Milnor fibration, and that the Milnor fiber $F = Q^{-1}(1)$ should be of interest. In [78] it was shown that this Milnor fibration is constant in a lattice-isotopic family, so that the Milnor fiber is indeed an invariant of lattice-isotopy. Because of this we propose the following definition, analogous to the definition made in the theory of knots.

Definition 4.3. *Two arrangements are called* (topologically) equivalent *if they are lattice-isotopic. We say the arrangements have the same* (topological) type.

Thus, arrangements are topologically equivalent if and only if they lie in the same path component of some matroid stratum in the Grassmannian. With this terminology, we have the following result.

Theorem 4.4. [78] *The Milnor fiber and fibration are invariants of topological type.*

Now, F is simply an $n-$fold cover of the complement of the projectivized arrangement in $\mathbb{C}P^{\ell-1}$. Since the algorithms of the previous section work to compute the fundamental group of this latter space, questions involving the fundamental group and cohomology of F are also questions involving the group of the arrangement. In particular, while the cohomology of M is determined by the intersection lattice, that of F may not be. The situation is analogous to that of plane curves, where work going back to Zariski [92] shows that not only the type but the position of the singularities affects the irregularity. (The irregularity here is simply half the "excess" in the first betti number of F.)

Early results concerning the Milnor fiber of an arrangement (often in the general context of plane curves) appear in work of Libgober [54, 51, 52, 53] and Randell [79], particularly with respect to Alexander invariants. Libgober's work gave considerable information about the homology of the Milnor fiber in relation to the

number and type of singularities of the arrangement, their position and the number of lines. The paper [79] observed that the Alexander polynomial was equal to the characteristic polynomial of the monodromy on the Milnor fiber.

The paper of Artal-Bartolo [2] included an interesting example: for the rank three braid arrangement A_3 the first betti number of the Milnor fiber is seven, an excess of two over the five "predicted" by the number of lines. (This result can be obtained as an interesting exercise by applying the Reidemeister-Schreier rewriting algorithm to the presentations of the fundamental group.) Orlik and Randell [62] showed that in the generic case the cohomology of the Milnor fiber is minimal, given by the number of lines, below the middle dimension.

Cohen and Suciu carry forward the study of the Milnor fiber in [11]. Using the group presentation and methods of Fox calculus they give twisted chain complexes whose homology gives that of the Milnor fiber. Their methods are effective, and several explicit examples are given. The monodromy action on the Milnor fiber is of course crucial, and this monodromy is determined as well.

Finally, we note the following problem, which remains open after many years.

Problem 4.5. *Prove that the homology of the Milnor fiber of $\mathcal{A}$ depends only on the underlying matroid.*

Acknowledgements . The idea to hold a birthday conference in honor of Peter Orlik was initially suggested by the second author in 1995. We would like to thank Mutsuo Oka And Hiroaki Terao for their hard work in organizing the meeting. We also wish to thank the referee for his helpful observations concerning deformations of arrangements.

The two authors were both students of Peter Orlik in Madison, Randell in the early 1970's and Falk in the early 1980's. We are happy to have the chance to thank him, in print, for introducing us to the field of hyperplane arrangements and for his enthusiasm, friendship and support through the years.

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